To summarize, therefore, if a given $\rho(z)$ is to be capable of physical realization as a cascade of lossless transmission lines terminated by a resistor the following conditions must be satisfied.

1) $\rho(z)$ must be a unit real.
2) $|\rho(\pm 1)| \neq 1$.
3) $\rho(0) \neq \rho(\infty)$.
4) The coefficients of the numerator and denominator polynomials of $\rho(z)$ must satisfy (5).

Conditions 1) and 4) above are both extensions of Young's necessary and sufficient conditions for the removal of a unit element with consequent reduction in the degree of $\rho(z)$ and they are in themselves sufficient. Conditions 2) and 3) are in the nature of tests which can be rapidly applied to eliminate forms of $\rho(z)$ impossible for the realization desired.

C. S. Gledhill
Dept. of Elec. Engrg. and Electronics
University of Manchester
Institute of Science and Technology
Manchester, England

Coupling to a Desired Wave

Technical situations often arise where a wave of given characteristics must be propagated from a transmitting antenna to a receiving antenna. An example is shown in Fig. 1, where a Goubau wave must be launched along a dielectric-clad metallic conductor. The problem then arises of finding out how efficiently the wave is launched, i.e., how much of the total radiated power is propagated in the desired form. In this correspondence we attempt to give a very general definition of this coupling coefficient. Figure 2 illustrates a typical structure in which a radiating system is bounded by a surface S_1 consisting of a conducting wall S'' and a radiating aperture S'. In particular, S' can be closed surface completely surrounding the sources $\bar{J}$. The desired wave is characterized by a tangential electric field $\bar{E}_{t1}$ on S'. The actual tangential field is $\bar{E}_t$, and our problem consists of splitting $\bar{E}_t$ as

$$\bar{E}_t = \lambda \bar{E}_{t1} + \bar{E}_{t2}. \qquad (1)$$

The complex number λ represents the "launching coefficient" of the desired wave. In the language of functional analysis, $\lambda\bar{E}_{t1}$ is the projection of the actual wave (uniquely determined by its $\bar{E}_t$) on the subspace formed by the desired wave. To separate the two parts, the desired wave $\lambda\bar{E}_{t1}$ should be orthogonal, in some sense, to the complementary wave $\bar{E}_{t2}$. It is therefore necessary to introduce a suitable definition of the scalar product. This definition should lead to a splitting such that the sum of the powers carried by the individual terms in (1) is equal to the total radiated power. As "power orthogonality" is involved, the definition of the scalar product must necessarily contain the tangential com-

Manuscript received August 22, 1966; revised November 28, 1966.

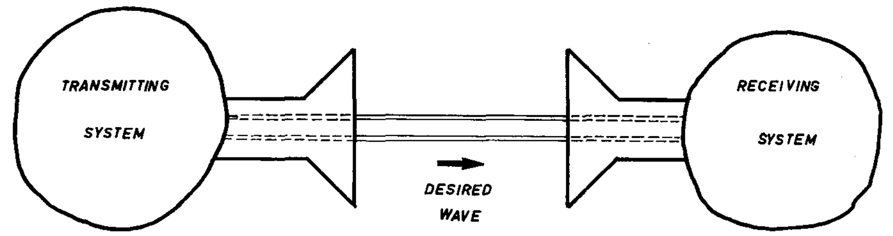

Fig. 1. Launching of a Goubau wave.

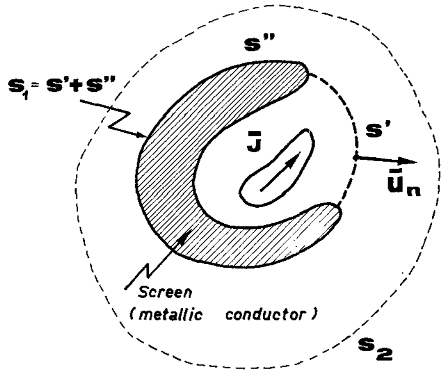

Fig. 2. A typical radiating system.

ponents of $\bar{E}$ and $\bar{H}$ on S', which can be collectively represented by a four-vector $\bar{\epsilon}$. A suitable definition for the product of waves whose four vectors are, respectively, $\bar{\epsilon}_a$ and $\bar{\epsilon}_b$ is

$$\langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle = \frac{1}{4} \iint_{S'} \bar{u}_n \cdot (\bar{E}_{ta}^* \times \bar{H}_{tb} + \bar{E}_{tb} \times \bar{H}_{ta}^*) dS. \qquad (2)$$

It is immediately apparent that the scalar product $\langle \bar{\epsilon}_a, \bar{\epsilon}_a \rangle$ of a wave with itself is the (real) average power radiated by the wave. It is also apparent that the scalar product of two waves depends, in general, on the boundary surface used to evaluate $\langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle$. Under certain circumstances, however, the value remains the same for two surfaces such as S_1 and S_2 in Fig. 2. This is true when:

1) the medium between S_1 and S_2 is Hermitian (condition: $\epsilon = \epsilon\dagger$ and $\mu = \mu\dagger$); example: a lossless plasma.
2) there are, in addition, no power sources between S_1 and S_2 (i.e., $\bar{J} = 0$ for both waves).

The property is easily proved by integrating

$$\begin{aligned}
&\operatorname{div} (\bar{E}_a^* \times \bar{H}_b + \bar{E}_b \times \bar{H}_a^*) \\
&= \bar{H}_b \cdot \operatorname{curl} \bar{E}_a^* - \bar{E}_a^* \cdot \operatorname{curl} \bar{H}_b + \bar{H}_a^* \\
&\quad \cdot \operatorname{curl} \bar{E}_b - \bar{E}_b \cdot \operatorname{curl} \bar{H}_a^* \\
&= \bar{H}_b \cdot (j\omega\mu^* \cdot \bar{H}_a^*) - \bar{E}_a^* \cdot (j\omega\epsilon \cdot \bar{E}_b) \\
&\quad + \bar{H}_a^* \cdot (-j\omega\mu \cdot \bar{H}_b) - \bar{E}_b \cdot (-j\omega\epsilon^* \cdot \bar{E}_a^*) \\
&= 0
\end{aligned}$$

between S_1 and S_2. Assume, in particular, that S_2 is the sphere at infinity. The fields on that sphere have the general form

$$\bar{E} = \bar{F} \frac{e^{-jkR}}{R}$$

and

$$\bar{H} = \frac{1}{R_0} (\bar{u}_R \times \bar{E})$$

with

$$R_0 = (120\pi)\Omega.$$

For such case,

$$\langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle = \frac{1}{2R_0} \iint \bar{F}_a^* \cdot \bar{F}_b d\Omega. \qquad (3)$$

This equation shows that $\langle \bar{\epsilon}_a, \bar{\epsilon}_a \rangle$ is not only real, but also non-negative when the external medium satisfies the conditions enunciated above. It also shows that fields which have power orthogonality on a given surface have radiation vectors $\bar{F}_a$ and $\bar{F}_b$ satisfying

$$\iint \bar{F}_a^* \cdot \bar{F}_b d\Omega = 0. \qquad (4)$$

Having thus explored some of the properties of the power scalar product, we are now in a position to determine λ in (1). Use of the orthogonality requirement for $\bar{\epsilon}_1$ and $\bar{\epsilon}_2 = \bar{\epsilon} - \lambda\bar{\epsilon}_1$ yields

$$\lambda = \frac{\langle \bar{\epsilon}_1, \bar{\epsilon} \rangle}{\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle}. \qquad (5)$$

It is to be noticed that the "desired wave" is determined to within a constant factor. To lift the indeterminacy, one often adopts the "unit power normalization" $\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle = 1$. It is now easy to show, with the help of the orthogonality property, that

$$\langle \bar{\epsilon}, \bar{\epsilon} \rangle = |\lambda|^2 \langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle + \langle \bar{\epsilon}_2, \bar{\epsilon}_2 \rangle. \qquad (6)$$

The total radiated power is, therefore, the sum of the powers in $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$, as required. The "power launching efficiency" is

$$\rho = \frac{|\lambda|^2 \langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle}{\langle \bar{\epsilon}, \bar{\epsilon} \rangle} = \frac{\langle \bar{\epsilon}_1, \bar{\epsilon} \rangle \langle \bar{\epsilon}, \bar{\epsilon}_1 \rangle}{\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle \langle \bar{\epsilon}, \bar{\epsilon} \rangle} \leq 1. \qquad (7)$$

These various quantities are normally dependent on the choice of the "launching" aperture S', unless the conditions stated above for the constancy of the scalar product are satisfied.

At this point, we have solved the problem of determining the amplitude of the desired wave, $\bar{\epsilon}$ being given. However, knowledge of $\bar{E}_t$ alone, and not of $\bar{\epsilon}$, should be sufficient to determine the field uniquely. Furthermore, $\bar{E}_t$ is normally unknown, and must be determined by use of the boundary conditions across S' combined with a knowledge of the sources inside and outside S_1. The resulting "coupled regions" problem is a very difficult one to solve. This formulation can be clarified by introducing[1] the concept of "admittance matrix" looking into a given region. This procedure does not necessarily facilitate the actual computational work, but establishes a welcome link with network theory. To this effect we introduce a complete set of real orthonormal vectors $\bar{\alpha}_m$ on S', and perform the expansions

$$\bar{E}_t = \sum V_m \bar{\alpha}_m$$

$$\bar{H}_t \times \bar{u}_n = \sum I_m \bar{\alpha}_m.$$

Between the V's and the I's exists the circuit relationship

$$\bar{I} = \mathcal{Y}_0 \cdot \bar{V} \qquad (8)$$

where $\mathcal{Y}_0$ is the radiating admittance looking outside S_1. The scalar product takes the form

$$\begin{aligned} \langle \bar{\epsilon}_a, \bar{\epsilon}_b \rangle &= \frac{1}{4} (\bar{V}_a^* \cdot \bar{I}_b + \bar{V}_b \cdot \bar{I}_a^*) \\ &= \frac{1}{4} (\bar{V}_a^* \cdot \mathcal{Y}_0 \cdot \bar{V}_b + \bar{V}_b \cdot \mathcal{Y}_0^* \cdot \bar{V}_a^*) \\ &= \frac{1}{4} \bar{V}_a^* \cdot (\mathcal{Y}_0 + \mathcal{Y}_0^\dagger) \cdot \bar{V}_b \\ &= \frac{1}{2} \bar{V}_a^* \cdot \mathcal{H}_0 \cdot \bar{V}_b \end{aligned} \qquad (9)$$

where $\mathcal{H}_0$ is the Hermitian part of $\mathcal{Y}_0$. When the medium outside S_1 is symmetric (i.e., $\epsilon = \tilde{\epsilon}$, $\mu = \tilde{\mu}$, the conductivity being incorporated in ϵ), $\mathcal{Y}_0$ turns out to be symmetric,[2] and its Hermitian part is the conductance matrix $\mathcal{G}_0$. In the absence of energy sources, $\langle \bar{\epsilon}_a, \bar{\epsilon}_a \rangle$ is always positive, hence, $\mathcal{H}_0$ is positive definite. To determine $\bar{V}$, two circuit equations of the type (8) must be written.[1] One obtains, by elimination of $\bar{I}$,

$$\bar{I}_g = (\mathcal{Y}_0 + \mathcal{Y}_i) \cdot \bar{V} \qquad (10)$$

where $\mathcal{Y}_i$ is the admittance looking inside S_1. The column vector $\bar{I}_g$ represents the expansion coefficients I_{gm} of the surface current produced by the sources $\bar{J}$ on the short-circuited surface S'. In other words, the current density $\bar{J}_s$ is equal to $\Sigma I_{gm} \bar{\alpha}_m$. Inversion of the $(\mathcal{Y}_0 + \mathcal{Y}_i)$ matrix gives

$$\bar{V} = (\mathcal{Y}_0 + \mathcal{Y}_i)^{-1} \cdot \bar{I}_g = \mathcal{Z} \bar{I}_g. \qquad (11)$$

To obtain optimum launching efficiency, this value of $\bar{V}$ should be proportional to the vector $\bar{V}_1$ corresponding to the desired tangential field $\bar{E}_{t1}$. Alternatively, the sources should induce an (optimum) wall-current density

$$\bar{I}_{g1} = (\mathcal{Y}_0 + \mathcal{Y}_i) \cdot \bar{V}_1 \qquad (12)$$

We can now express λ in the following form

$$\begin{aligned} \lambda &= \frac{\langle \bar{\epsilon}_1, \bar{\epsilon} \rangle}{\langle \bar{\epsilon}_1, \bar{\epsilon}_1 \rangle} = \frac{\bar{V}_1^* \cdot \mathcal{H}_0 \cdot \bar{V}}{\bar{V}_1^* \cdot \mathcal{H}_0 \cdot \bar{V}_1} \\ &= \frac{(\mathcal{Z}^* \cdot \bar{I}_{g1}^*) \cdot \mathcal{H}_0 \cdot (\mathcal{Z} \cdot \bar{I}_g)}{(\mathcal{Z}^* \cdot \bar{I}_{g1}^*) \mathcal{H}_0 \cdot (\mathcal{Z} \cdot \bar{I}_{g1})} \\ &= \frac{\bar{I}_{g1}^* \cdot [\mathcal{Z}^\dagger \cdot \mathcal{H}^\dagger \cdot \mathcal{Z}] \cdot \bar{I}_g}{\bar{I}_{g1}^* \cdot [\mathcal{Z}^\dagger \cdot \mathcal{H}_0 \cdot \mathcal{Z}] \cdot \bar{I}_{g1}}. \end{aligned} \qquad (13)$$

[1] J. Van Bladel, "The matrix formulation of scattering problems," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-14, pp. 130–135, March 1966.

[2] J. Van Bladel, "A generalized reciprocity theorem for radiating apertures," *Arch. Elekt. Übertragung*, pp. 447–450, August 1966.

Equation (13) shows, in principle at least, how to determine the coupling coefficient λ, given the actual short-circuit current $\bar{I}_g$ and its optimum value $\bar{I}_{g1}$. This relationship can also be put in the form of a splitting

$$I_g = \lambda \bar{I}_{g1} + \bar{I}_{g2} \qquad (14)$$

where the two parts, $\lambda \bar{I}_{g1}$ and $\bar{I}_{g2}$, are orthogonal with respect to a scalar product

$$\langle \bar{I}_{ga}, \bar{I}_{gb} \rangle = \bar{I}_{ga}^* \cdot [\mathcal{Z}^\dagger \cdot \mathcal{H}_0 \cdot \mathcal{Z}] \cdot \bar{I}_{gb} \qquad (15)$$

It is to be noticed that the bracketed matrix is Hermitian.

J. Van Bladel
Laboratory for Electromagnetism
and Acoustics
University of Ghent
Ghent, Belgium

The Numerical Solution of TEM Mode Transmission Lines with Curved Boundaries

Introduction

A finite difference solution of Laplace's equation has been used by several authors [1], [2] to obtain the transmission-line parameters of uniform TEM transmission lines with straight-line boundaries. This correspondence shows that useful results can also be obtained where the finite difference solution is modified to permit curved boundaries.

Theory

Figure 1 shows a finite difference net used to obtain the odd mode transmission line parameters of the transmission line with the cross section shown in Fig. 2. It is seen that four different types of nodes are produced by the net. They are:

a) Conducting nodes such as A
b) Irregular nodes such as B
c) Exterior nodes such as C
d) Open-circuit nodes such as D
e) Interior nodes such as E.

Only the irregular nodes are different from those treated previously [1]. A suitable finite difference approximation for use at irregular nodes is given by Forsythe and Wasaw [3] and Weber [7]:

$$\left(\frac{1}{h_1 h_2} + \frac{1}{h_3 h_4} \right) V_0 = \frac{V_1}{h_1(h_1 + h_3)} + \frac{V_2}{h_2(h_2 + h_4)} + \frac{V_3}{h_3(h_1 + h_3)} + \frac{V_4}{h_4(h_2 + h_4)}$$

where the notation is given in Fig. 3.

Although the error in this approximation is of the order $0(h)$ as compared with $0(h^2)$ for interior nodes it has been shown [4] that the overall accuracy is still of the order $0(h^2)$.

Manuscript received August 29, 1966; revised November 15, 1966.

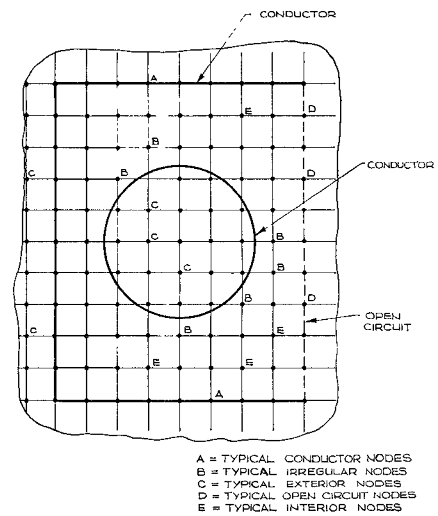

Fig. 1. Finite difference net.

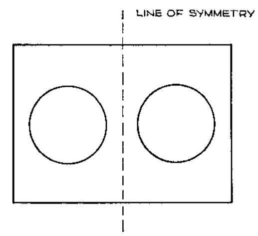

Fig. 2. Transmission line cross section.

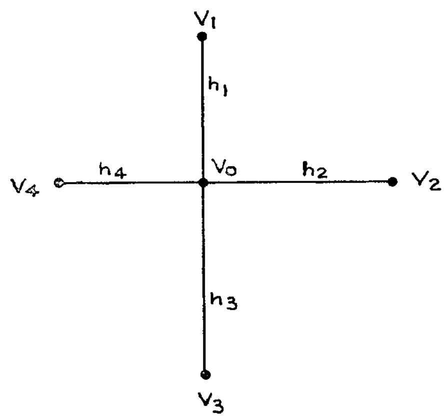

Fig. 3. Notation for irregular nodes.

Programming

To analyze a particular problem the program should only require the boundaries to be specified. This was achieved by scanning all the nodes of the net and suitably tagging each irregular node. All conductor and exterior nodes were also tagged by putting them at an integral value of potential. Successive over-relaxation [1] was used to obtain the approximate potential at each node of the net. The transmission-line parameters were obtained from either Gauss's theorem [1] or the energy [6] which was obtained by interpolating an approximate but continuous potential function satisfying the boundary values.